Leibniz Universität Hannover
Wirtschaftswissenschaftliche Fakultät
Institut für Produktionswirtschaft
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Seat number:

Summer semester 2024
Exam for the lecture
“Operations Research”

Please note:

- The exam consists of **13** pages (including this cover sheet). Please check your copy for completeness and ask for a new one in case of missing pages.
- All exam tasks are mandatory and have to be processed.
- Within the exam's time frame of 60 minutes 60 points are potentially attainable.
- Your solution approach has to be comprehensible. Show your work!
- Permitted items and materials are a non-programmable calculator and a handwritten two-sided standard DIN A4 or letter format sheet with formulas, notes etc. of your choice. You may also use an English language dictionary.
- You will find enough space in the exam sheets to answer the questions. Do not use your own paper and do not disassemble your exam copy.
- Enter your personal data first.
- You may answer the questions using either the German or the English language.

Personal data:

Family name	Given name	Matr. number	Study program	Semester of study

Grading results:

Problem	1	2	3	4	5	Sum
Points						

1. **Improvement search: (10 P.)**

   We consider real-valued variables w_1 and w_2.

   (a) Assume we are minimizing the function $f(w_1, w_2) = 3w_1^2 w_2$.

      i. Construct at point $\mathbf{w} = (w_1, w_2) = (2, 3)$ an improving direction from the gradient of this function. (2 P.)

      ii. Determine by an appropriate gradient test whether at point $\mathbf{w} = (w_1, w_2) = (1, 2)$ the direction $\Delta\mathbf{w} = (-2, 1)$ improves on the objective function. (2 P.)

(b) Explain briefly what characterizes in an improving search an "active constraint" and what a "feasible direction". (2 P.)

(c) Draw an example of a feasible set for two continuous variables w_1 and w_2 where an improving search algorithm cannot be guaranteed to find an optimal solution due to this feasible set and explain your reasoning! (2 P.)

(d) Consider for $w_1, w_2 \in \mathbb{R}$ the maximization of the objective function $g(w_1, w_2) = w_1 \cdot \sin(w_1) + 0 \cdot w_2$ and explain whether for this objective function, the improving search algorithm can find a global maximum! (2 P.)

2. **Branch & Bound** **(15 P.)**

Consider the following integer linear program with a **minimization** objective function:

$$
\begin{aligned}
&\text{Min } 4x_1 + 6x_2 \\
&\text{s.t.} \\
&2x_1 + 12x_2 \geq 23 \\
&10x_1 + 2x_2 \geq 27 \\
&x_i \in \{0, 1, 2, 3, 4, ...\}, \qquad\qquad i \in \{1, 2\}
\end{aligned}
$$

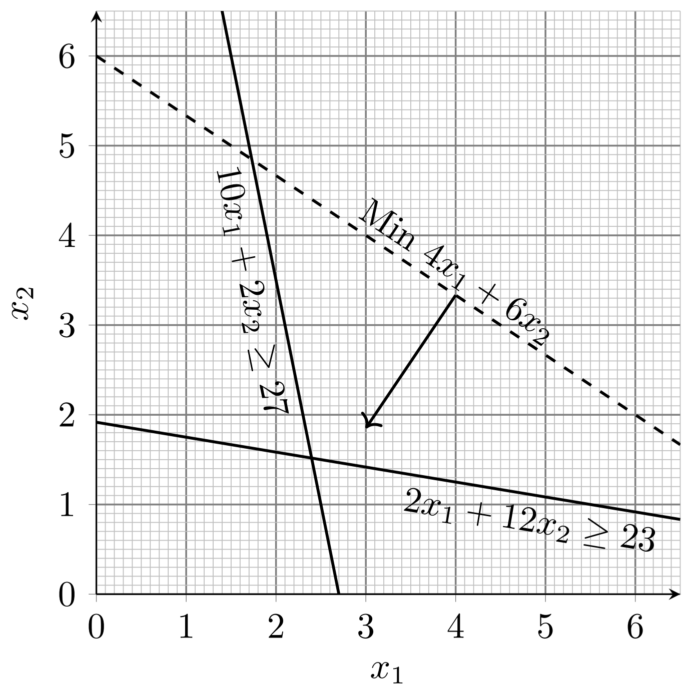
Determine the optimal solution of this integer **minimization** problem by applying a branch&bound algorithm according to the the following specifications:

- Perform a depth-first search!
- Start with incumbent solution ($x_1 = 5,\ x_2 = 5$).
- If in the linear programming relaxation of a candidate problem both variables x_1 and x_2 should be fractional, break ties in favor of the first variable, i.e., branch on x_1.
- When branching on a fractional variable x_i by creating two new problems, create the new problem to be examined next in the depth-first search by rounding up and the other new problem (to be examined later in the depth-first search) by rounding down.
- Number the problems according to the sequence in which you determine their relaxations and make branching or bounding decisions in the decision tree based on the solution of the candidate problem.

Hints and tasks:

- You can determine the values of relaxed variables x_1 and x_2 in a relaxation of a candidate problem to a sufficient degree of accuracy by modifying and reading from the figure on the following page. On this basis you can compute to a sufficient degree of accuracy the objective function values as well.
- Document in your search tree for each candidate the relaxation outcome, consequences for lower and upper bounds, and the resulting decision.
- Give the optimal solution and the optimal objective function value.

The following figure (see next page) shows the visualization of the objective function and the constraints:



$10x_1 + 2x_2 \geq 27$
Min $4x_1 + 6x_2$
$2x_1 + 12x_2 \geq 23$
x_1
x_2
0
1
2
3
4
5
6

3. **Linear Programming** **(14 P.)**

Consider the following linear program (LP), denoted as the "primal" problem:

$$\begin{aligned} \text{Max } & 5x_1 + 3x_2 \\ \text{subject to} & \\ x_1 & \leq 2 \\ x_2 & \leq 3 \\ x_1 + x_2 & \leq 10 \\ x_1, x_2 & \geq 0 \end{aligned}$$

(a) Determine by inspection

  i. the optimal solution to the problem, (2 P.)

  ii. the objective function value for the optimal solution, (1 P.)

  iii. the values of the dual variables in the optimal solution. (2 P.)

(b) Using your results, show that for all three constraints the primal complementary slackness condition holds! (3. P)

(c) Determine the dual program corresponding to the primal LP! (2 P.)

(d) Give the standard form of the primal LP! (2 P.)

(e) Does the standard form version of the primal LP possess variables which can never be non-basic? If so, give those variable(s). In either case, justify your answer! (2 P.)

4. **Integer and binary optimization** **(3 P.)**

Assume that you are dealing with a linear integer or binary program. Can the improving search algorithm be used to solve that program? Explain your reasoning!

5. **Models and Modeling:** **(18 P.)**

(a) Write down algebraically an example of a linear programming model with two decision variables x_1 and x_2 which is feasible and possesses exactly one feasible solution with respect to those variables x_1 and x_2. Give the values for x_1 and x_2 in that solution! Does finding this solution pose a difficulty when applying the rudimentary simplex search algorithm 5A to that model? Explain! (4 P.)

(b) Draw graphically an example of a linear program with two decision variables x_1 and x_2 in which multiple different (!) basic solutions of the model in standard form correspond to the same optimal solution vector $x = (x_1^*, x_2^*)$. (4 P.)

(c) Write down an example of a small or even tiny feasible non-linear program and give a hint as to why it is non-linear! (4 P.)

(d) Consider the following binary program:

$$\begin{aligned}&\text{Maximize } 10x_1 + 10x_2 + 10x_3 + 10x_4\\&\text{subject to}\\&40x_1 + 50x_2 + 60x_3 + 70x_4 \leq 150\\&x_1, x_2, x_3, x_4 \in \{0, 1\}\end{aligned}$$

For this program

i. determine by inspection the optimal solution and give the objective function value for this solution, (2 P.)

ii. determine by inspection the optimal solution of the linear programming relaxation and give the objective function value for this linear programming relaxation, (2 P.)

iii. and explain how and why a Branch&Bound algorithm to solve such a binary program would proceed in the root node of the search tree, given this optimal solution of the linear programming relaxation of the original problem. (2 P.)